12.) Space charge

Concerning beam transport we did not include the forces between the charged particles due to electromagnetic interaction. For high beam currents the space charge of the beam will be significant and space charge forces cannot be neglected any longer. The space charge forces lead to a beam defocusing in particular for ion beams in the Low Energy Beam Transport (LEBT) section.

The space charge effects are called collective effects. In addition, the electromagnetic interaction of the charged beam particles with the beam pipes and vacuum chambers has to be taken into account (mirror charges and currents).

Collective effects can be divided into two distinct groups according to the physics involved. The compression of a large number of charged particles into a small volume increases the probability for collisions of particles within the same beam. Because particles perform oscillations in the beam transport, statistical collisions occur in longitudinal, as well as transverse phase space often causing a mixing of phase space coordinates.
The other group of collective effects includes effects which are associated with electromagnetic fields generated by the collection of all particles in a beam.

ß small → space charge forces dominate

ß ~ 1 → particle wall interaction dominates → wake fields

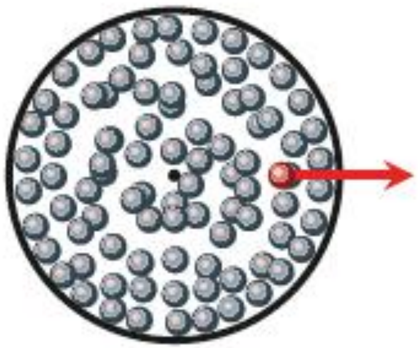
If wake fields are amplified via imperfections → collective instabilities

12.1. Space charge effects – non relativistic

We consider two particles with identic charge q.
The repelling force is defined by the Coulomb-force.

$$F_C = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q^2}{r^2}$$

Now we consider a test particle in a cylinder symmetric beam with the charge q_i.

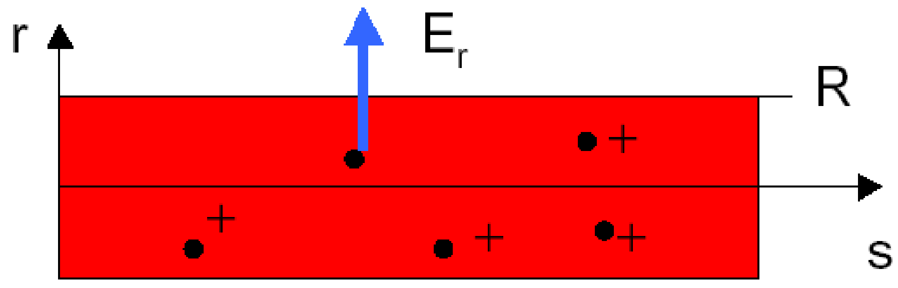


The Coulomb force pushes the test particle outwards. The force is zero on axis and is increasing towards the beam edge. This radial force leads to a beam defocusing. The space charge field can be derived by:

$$div\vec{E} = -\frac{\rho}{\varepsilon_0} \quad \Rightarrow \quad \int_F \vec{E} \cdot d\vec{F} = -\int_V \frac{\rho}{\varepsilon_0} dV$$

$$E_r(r) = -\frac{1}{2\pi\varepsilon_0 \cdot rl} \int_V \rho \cdot dV$$

$$E_r(r) = -\frac{1}{\varepsilon_0 r} \int_0^r \rho(r') \cdot r' dr'$$



We assume cylinder symmetry and no variation in s-direction. In case of a homogeneous space charge distribution within the beam we have $\rho(r') = \rho_0$ and we get

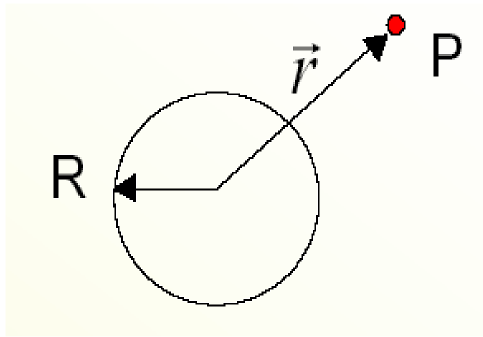
A: r < R $$\Rightarrow E(r) = -\frac{\rho_0 \cdot r}{2\varepsilon_0}$$

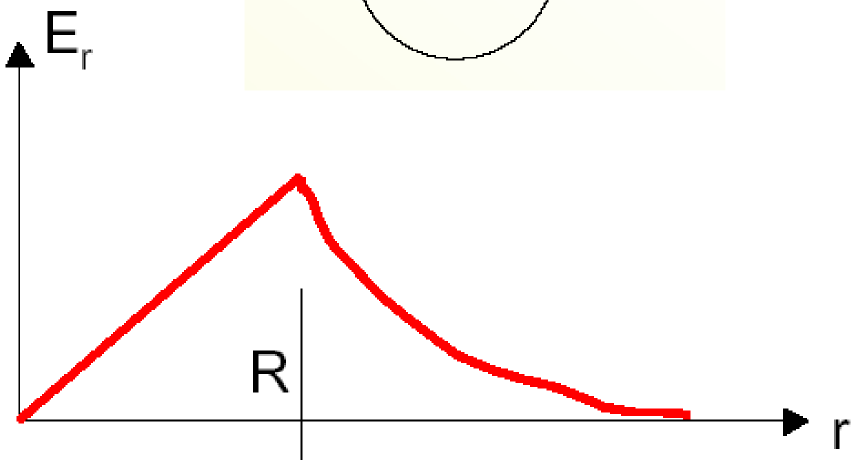
B: r > R $$\Rightarrow E(r) = -\frac{\rho_0}{2\varepsilon_0}\frac{R^2}{r}$$

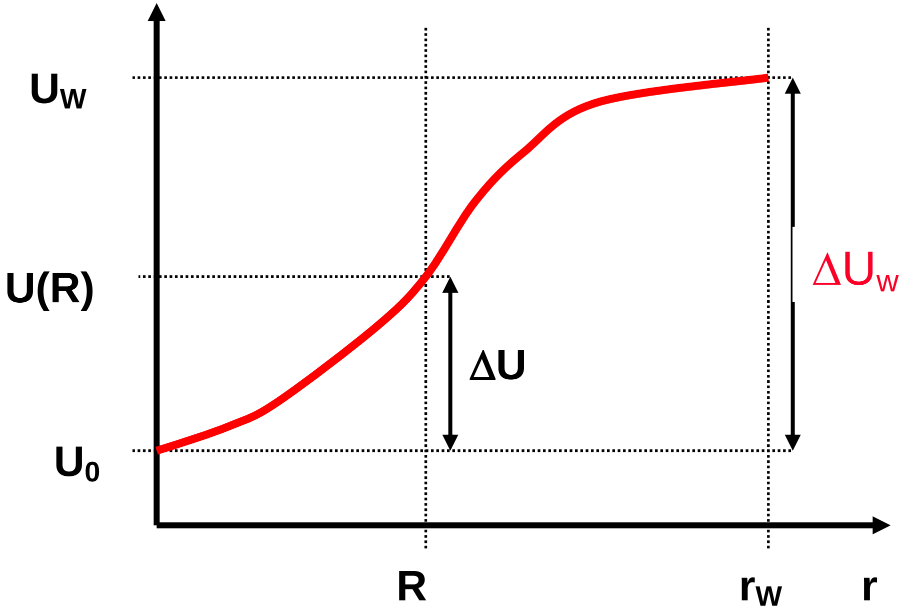
So we can derive the potential:

$$U(r) = -\int_0^r E(r)dr + U_0$$

A: r < R $$\Rightarrow U(r) = \frac{\rho_0 \cdot r^2}{4\varepsilon_0} + U_0$$







B: r > R $$\Rightarrow U(r) = \frac{\rho_0 \cdot R^2}{2\varepsilon_0} \ln\left(\frac{r}{R}\right) + U(R) = \frac{\rho_0 \cdot R^2}{4\varepsilon_0}\left[2 \cdot \ln\left(\frac{r}{R}\right) + 1\right] + U_0$$

We get $U_0 = U(r=0)$. U_0 we can calculate by using the known wall potential $U_W = U(r_W)$:

$$U_W = \frac{\rho_0 \cdot R^2}{4\varepsilon_0}\left[2 \cdot \ln\left(\frac{r_W}{R}\right) + 1\right] + U_0$$

Thus we derive for the potential depression between wall and beam axis

$$\Delta U_W = U_W - U_0 = \frac{\rho_0 \cdot R^2}{4\varepsilon_0}\left[2 \cdot \ln\left(\frac{r_W}{R}\right) + 1\right]$$

and between beam edge and beam axis $$\Delta U = U(R) - U_0 = \frac{\rho_0 \cdot R^2}{4\varepsilon_0}.$$

If the velocity modulation of the beam particles by the space charge potential is small, the space charge density stays homogeneous

$$j = \rho_0 \cdot v_z = \frac{I}{\pi \cdot R^2} \quad \Rightarrow \quad \rho_0 = \frac{I}{\pi \cdot R^2 v_s} = \frac{I}{\pi \cdot R^2 \sqrt{\frac{2qU_{acc}}{m}}}$$

U_{acc} = Acceleration voltage for the beam particles.

Finally we get:

A: r < R
$$U(r) = \frac{I}{4\pi\varepsilon_0} \frac{r^2}{R^2 \sqrt{\frac{2qU_{acc}}{m}}} + U_0$$

B: r > R
$$U(r) = \frac{I}{4\pi\varepsilon_0} \frac{1}{\sqrt{\frac{2qU_{acc}}{m}}} \left[2 \cdot \ln\left(\frac{r}{R}\right) + 1 \right] + U_0$$

For a more realistic investigation the Poisson equation in cylindrical coordinates has to be solved.

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial U(r)}{\partial r} \right) = \frac{I}{\pi\varepsilon_0 R^2 \sqrt{\frac{2qU(r)}{m}}}$$

$$U(r)=\left(\frac{9}{16\pi\varepsilon_0\sqrt{\frac{2q}{m}}}\right)^{2/3} I^{2/3}\left(\frac{r}{R}\right)^{4/3}$$

In case the space charge is very high, beam particles can be reflected. This effect is called virtual cathode limit. This is dominant for low beam energies. Therefore a maximum current for each acceleration voltage can be transported. For *r*=*R* we get

$$\frac{U^{3/2}}{I_{\max}}=\frac{9}{16\pi\varepsilon_0\sqrt{\frac{2q}{m}}}=P_{\max}^{-1}$$

(maximum Perveance P_{max})

10.2. Space charge effects – relativistic

For relativistic ion beams:

$$E(r)=\frac{\rho_0\cdot r}{2\varepsilon_0}$$

and

$$\rho_0=\frac{I}{\pi R^2\beta\cdot c}=\frac{I}{\pi R^2 c\sqrt{1-\frac{1}{\gamma^2}}}$$

The self-magnetic field results from $rot\vec{B} = \mu_0 \vec{j}$ or $\int_0^{2\pi} B_\phi r d\phi = \mu_0 j \oint dF$

$$\Rightarrow 2\pi B_\phi r = \mu_0 j \pi \cdot r^2$$

$$\Rightarrow B_\phi(r) = \frac{\mu_0}{2} j \cdot r = \frac{\mu_0 \rho_0}{2} \beta c \cdot r$$

The space charge forces depend on the velocity of the particles. Aside the electric field the beam creates an azimuthal magnetic field. These fields create a radial component of the Lorentz force:

$$F_r(r) = q\left(E_r - vB_\phi\right) = q\left(\frac{\rho_0 \cdot r}{2\varepsilon_0} - \frac{\mu_0 \rho_0}{2} v^2 \cdot r\right) = q\frac{\rho_0}{2\varepsilon_0}\left(1 - \beta^2 c^2 \mu_0 \varepsilon_0\right) \cdot r$$

With $c^2 = \frac{1}{\mu_0 \varepsilon_0}$ we get

$$F_r(r) = q\frac{\rho_0}{2\varepsilon_0}\left(1 - \beta^2\right) \cdot r = \frac{q\rho_0}{2\varepsilon_0 \gamma^2} \cdot r$$

The space charge defocusing is a non-relativistic effect. The self-magnetic field compensates for relativistic energies the repulsive electric forces of the space charge. In case of space charge compensation (f < 1 – degree of compensation)

$$E(r)=\frac{\rho_0\cdot r}{2\varepsilon_0}(1-f) \quad \Rightarrow \quad F_r(r)=q\frac{\rho_0}{2\varepsilon_0}\left(1-f-\beta^2\right)\cdot r$$

Thus the result can be a net focusing of the beam (relativistic self-focusing) due to the space charge compensation:

In case of a drift the equation of motion is:

$$F_r(r)=m\gamma\cdot\ddot{r}=m\gamma(\beta c)^2 r''=q\frac{\rho_0}{2\varepsilon_0}\left(1-f-\beta^2\right)\cdot r=\frac{q\rho_0}{2\varepsilon_0\gamma^2}\left(1-\gamma^2 f\right)\cdot r$$

$$r''-\frac{q\rho_0}{2\varepsilon_0 m(\beta c)^2\gamma^3}\left(1-\gamma^2 f\right)\cdot r=0$$

$$\rho\cdot v=j=\frac{I}{\pi a^2} \quad \Rightarrow \quad \rho=\frac{I}{\beta c\pi a^2}$$, a = beam radius

We now introduce the characteristic current (Alfvén current)

$$I_0=\frac{4\pi\varepsilon_0 mc^3}{q}=\frac{1}{30}\frac{mc^2}{q}$$

For electrons we get I_0 = 17045 A, for ions we get $I_0 = 3*10^7*A/q$ A.

$$r'' - \frac{q \cdot I}{2\pi\varepsilon_0 mc^3\beta^3\gamma^3 a^2}\left(1-\gamma^2 f\right)\cdot r = 0$$

$$\Rightarrow \qquad r'' - \frac{2I}{I_0\beta^3\gamma^3 a^2}\left(1-\gamma^2 f\right)\cdot r = 0$$

The characteristic current is used when the space charge density ρ_0 is replaced by the beam current. (Alfvén, H., Phys. Rev. 55 (1939) 425)
This current is a quasi-scaling of the space charge potential, hence Lawson introduced the generalized perveance in 1958:

$$K = \frac{2I}{I_0\beta^3\gamma^3}\left(1-\gamma^2 f\right) \qquad (*)$$

(J. D. Lawson, J. Electron Control, 5 (1958) 146)

Therefore in a drift we have

$$r'' - \frac{K}{a^2} r = 0$$

The space charge acts like a continuously defocusing lens, if the beam is not partially compensated with $-K/a^2$. With a quadrupole lens we get the following Hill DGL:

$$x'' + k_x x - \frac{K}{a^2} x = 0 \quad \text{und} \quad y'' + k_y y - \frac{K}{a^2} y = 0$$

with $$k_x = -k_y = k = \frac{g}{B \cdot \rho} = \frac{\text{gradient}}{\text{magnetic rigidity}} \; .$$

We have now the equation of motion for a single particle in case the phase spaces are not coupled (which is not the case for space charge forces). We now evaluate a distribution of particles in a beam.

$$x'' + k_x x - \frac{K}{a^2} x = x'' + k_x x - F_{sc} = 0$$

We now investigate the motion of the second moments (RMS-values) of the particle distribution:

$$\overline{x^2} = \frac{1}{n} \sum_{i=1}^{n} x_i^2 \quad , \quad \overline{xx'} = \frac{1}{n} \sum_{i=1}^{n} x_i x_i' \quad , \quad \overline{x'^2} = \frac{1}{n} \sum_{i=1}^{n} x_i'^2$$

$$\varepsilon_{RMS}^2 = \overline{x^2} \cdot \overline{x'^2} - \overline{xx'}^2$$

$$\frac{d}{ds} \overline{x^2} = \frac{1}{n} \sum_{i=1}^{n} \frac{d}{ds} \left(x_i^2 \right) = \frac{1}{n} \sum_{i=1}^{n} 2 x_i x_i' = 2 \overline{xx'}$$

$$\frac{d}{ds}xx' = \frac{1}{n}\sum_{i=1}^{n}\frac{d}{ds}\left(x_i x_i'\right) = \frac{1}{n}\sum_{i=1}^{n}\left(x_i' x_i' + x_i x''\right) = \overline{x_i'^2} + \overline{xx''}$$

$$\frac{d}{ds}\overline{x'^2} = \frac{1}{n}\sum_{i=1}^{n} 2x_i' x_i'' = 2\overline{x'x''}$$

By introducing the RMS-beam size $a = \sqrt{\overline{x^2}}$ we get

$$\frac{da^2}{ds} = 2aa' = \frac{d}{ds}\left(\overline{x^2}\right) = 2\overline{xx'} \quad \Rightarrow \quad aa' = \overline{xx'}$$

$$\frac{d}{ds}\left(aa'\right) = a'^2 + aa'' = \overline{x'^2} + \overline{xx''} \qquad \Rightarrow \quad a'' + \frac{a'^2 - \overline{x'^2}}{a} - \frac{\overline{xx''}}{a} = 0$$

$$\Rightarrow \quad a'' + \frac{a'^2 - \overline{x'^2}}{a} - \frac{\overline{x\left(-k_x x + F_{sc}\right)}}{a} = 0$$

$$a'' + \frac{a'^2 - \overline{x'^2}}{a} - k_x a - \frac{\overline{xF_{sc}}}{a} = a'' + \frac{a^2 a'^2 - a^2\overline{x'^2}}{a^3} - k_x a - \frac{\overline{xF_{sc}}}{a} = 0$$

$$a'' - \frac{\varepsilon_x}{a^3} + k_x a - \frac{\overline{xF_{sc}}}{a} = 0$$ (**)

This is the RMS-envelope equation. The term $\frac{\varepsilon_x}{a^3}$ is the emittance term. The emittance term is negative as it represents the divergence of the particle. The focusing term k_x*a must be positive.

The last term $\frac{\overline{xF_{sc}}}{a}$ is repulsive, as it represents the space charge action. The equation of motion of the particle distribution (**) is quite similar to the equation of motion of a single particle (*). The difference is the emittance.

$$\overline{xF_{sc}} = \frac{1}{n}\sum_{i=1}^{n} x_i \frac{K}{a^2} x_i = \frac{K}{a^2}\frac{1}{n}\sum_{i=1}^{n} x_i^2 = \frac{K}{a^2}\overline{x^2} = K$$

$$a'' + k_x a - \frac{\varepsilon_x}{a^3} - \frac{K}{a} = 0$$

The space charge forces are non-linear and couple the degrees of freedom (transverse and longitudinal). The space charge forces lead to RMS-emittance growth, because the transverse oscillations of the beam particles depend on the amplitudes (non-linear forces) and on the coordinates of the other directions (coupling).