9) Beam focusing and transport

Beam transport in axial symmetric fields

For the beam formation and transport in electron guns and ion sources, axial symmetric geometries are most often used. Therefore the equation of motion is transferred to cylindrical coordinates. In electromagnetic fields the motion of charged particles is driven by the Lorentz-force:

$$m \cdot \ddot{\vec{r}} = \vec{F}_L = q(\vec{E} + \vec{v} \times \vec{B})$$

In cylindrical coordinates (see literature):

$$\begin{aligned} m \cdot (\ddot{r} - r\dot{\theta}^2) &= q(E_r + r\dot{\theta} \cdot B_z - \dot{z} \cdot B_\theta) \\ m \cdot (r\ddot{\theta} + 2\dot{r}\dot{\theta}) &= q(E_\theta + \dot{z} \cdot B_r - \dot{r} \cdot B_z) \\ m \cdot \ddot{z} &= q(E_z + \dot{r} \cdot B_\theta - r\dot{\theta} \cdot B_r) \end{aligned} \quad (9.1a)$$

In axial symmetric fields: $E_\theta = B_\theta = 0$. In case of relativistic beam m is $m\gamma$ and $z = \beta \cdot c = v$

For the ion source regime it does not matter. So we get

$$\begin{aligned} r: \quad & m \cdot (\ddot{r} - r\dot{\theta}^2) = q(E_r + r\dot{\theta} \cdot B_z) \\ \theta: \quad & m \cdot (r\ddot{\theta} + 2\dot{r}\dot{\theta}) = q(\dot{z} \cdot B_r - \dot{r} \cdot B_z) \\ z: \quad & m \cdot \ddot{z} = q(E_z - r\dot{\theta} \cdot B_r) \end{aligned} \quad (9.1b)$$

For θ we get $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 2\dot{r}\dot{\theta} + r\ddot{\theta}$

$$\Rightarrow \quad m\cdot(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = \frac{m}{r}\frac{d}{dt}(r^2\dot{\theta}) = q(\dot{z}\cdot B_r - \dot{r}\cdot B_z)$$, with $m\cdot r^2\cdot\dot{\theta} = l$

l is the angular momentum of the particle around the beam axis. For the vector potential in the case of an axial symmetric magnetic field it is imperative:

$$\vec{B} = rot\,\vec{A}, \quad B_\theta = 0 \quad \Rightarrow \quad \frac{\partial A_r}{\partial z} = \frac{\partial A_z}{\partial r} \quad \forall A_r, A_z \quad \Rightarrow \quad A_r = A_z = 0$$

$$\vec{A} = (0, A_\theta, 0) \quad \Rightarrow \quad \vec{B} = (-\frac{\partial A_\theta}{\partial z}, 0, \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta))$$

The formula above can then be written as:

$$\Rightarrow \quad \frac{d}{dt}(r^2\dot{\theta}) = \frac{q}{m}r(\dot{z}\cdot B_r - \dot{r}\cdot B_z) = \frac{q}{m}r(-\dot{z}\cdot\frac{\partial A_\theta}{\partial z} - \frac{\dot{r}}{r}\cdot\frac{\partial(rA_\theta)}{\partial r})$$

$$= -\frac{q}{m}(\dot{z}\cdot\frac{\partial(rA_\theta)}{\partial z} + \dot{r}\cdot\frac{\partial(rA_\theta)}{\partial r}) = -\frac{q}{m}(\frac{d}{dt}(rA_\theta) - \frac{\partial}{\partial t}(rA_\theta))$$

Since we only consider static focussing fields the partial derivative with respect to time is zero, we get:

$$\frac{d}{dt}(mr^2\dot{\theta}+q\cdot r\cdot A_\theta)=0 \quad\Rightarrow\quad mr^2\dot{\theta}+q\cdot r\cdot A_\theta = const. = P_\theta$$

P_θ is generalized momentum of the θ-component. For the magnetic flux it is:

$$\Phi = \int_F \vec{B}\cdot d\vec{F} = \int_F rot\vec{A}\cdot d\vec{F} = \oint_{(F)} \vec{A}\cdot d\vec{s} \quad\rightarrow\quad \Phi(r) = \int_0^{2\pi} rA_\theta d\theta = 2\pi\, rA_\theta$$

Therewith the generalized momentum results in:

$$P_\theta = P_{\theta_0} = mr^2\dot{\theta}+\frac{q\cdot\Phi(r)}{2\pi} = const. \quad\Rightarrow\quad \dot{\theta} = \frac{r_0^2}{r^2}\dot{\theta}_0 + \frac{q}{2\pi\cdot m\cdot r^2}(\Phi_0(r_0)-\Phi(r)) \qquad (9.2a)$$

This is the so-called **Busch-Theorem** (angular momentum conservation)!

For a solenoid field one gets: B_z = const → $\Phi(r) = B_z\cdot\pi\cdot r^2$

Therefore the Busch-theorem results in:

$$\Rightarrow\quad \dot{\theta} = \frac{r_0^2}{r^2}\dot{\theta}_0 + \frac{q}{2m}\left(\frac{r_0^2}{r^2}B_{0z} - B_z\right) \qquad (9.2b)$$

Are the angular speed $\dot{\theta}_0$ and the magnetic field B_{0z} at the starting point zero, then

$\dot{\theta} = -\frac{q}{2m} B_z$ and therewith identical with the Lamor-frequency.

If particles from a field-free zone are confined by a magnetic field, they gyrate with the Lamor-frequency $\omega_L = \omega_c/2$.

Therefore the equation of motion for $\dot{\theta}_0 = 0$ is

$$
\begin{aligned}
&r: && m \cdot (\ddot{r} - r\dot{\theta}^2) = q(E_r + r\dot{\theta} \cdot B_z) \\
&\theta: && \dot{\theta} = \frac{q}{2m}\left(\frac{r_0^2}{r^2} B_{0z} - B_z\right) \qquad (9.3) \\
&z: && m \cdot \ddot{z} = q(E_z - r\dot{\theta} \cdot B_r)
\end{aligned}
$$

If we use the middle expression for the two others we gain the so called **beam envelope equation**

$$
\ddot{r} = \frac{q}{m} E_r + \left(\frac{q}{2m}\right)^2 \frac{r_0^4 B_{0z}^2}{r^3} - \left(\frac{q}{2m}\right)^2 B_z^2 r \qquad (9.4)
$$

The first term describes the force of the space charge, the second the centrifugal force or magnetic emittance and the third term stands for the focusing force of the magnetic field. For the z-dependence one gets:

$$m \cdot \ddot{z} = qE_z - qr\frac{q}{2m}\left(\frac{r_0^2}{r^2}B_{0z} - B_z\right) \cdot B_r \qquad (9.5)$$

$$\frac{d}{dt} = \frac{d}{dz} \cdot \frac{dz}{dt} = v_z \frac{d}{dz} \quad , \qquad \frac{d^2}{dt^2} = \frac{d}{dt}\left(v_z \frac{d}{dz}\right) = \dot{v}_z \frac{d}{dz} + v_z^2 \frac{d^2}{dz^2}$$

$$\ddot{r} = \frac{d^2 r}{dt^2} = \dot{v}_z \frac{dr}{dz} + v_z^2 \frac{d^2 r}{dz^2} = \dot{v}_z r' + v_z^2 r'' \qquad (9.6)$$

If one wants to display the motion of particles at the beam edge in dependence on z, one has to combine (9.4), (9.5) and (9.6):

$$v_z^2 r'' + \left[\frac{q}{m}E_z - \frac{q^2}{2m^2} r \left(\frac{r_0^2}{r^2}B_{0z} - B_z\right) B_r\right] \cdot r' = \frac{q}{m}E_r + \left(\frac{q}{2m}\right)^2 \frac{r_0^4 B_{0z}^2}{r^3} - \left(\frac{q}{2m}\right)^2 B_z^2 r \qquad (9.7)$$

This is the **paraxial equation**. If $E_z = 0$ and $B_r << B_z$, then r' can be neglected. B_r promotes a change in v_z and therewith in $W_{kin||}$. This means that B_r converts kinetic energy from the longitudinal into the transverse direction, a magnetic mirror effect.

The paraxial ray equation including space charge (affects only E_r) is then:

$$v_z^2 r'' + \left[\frac{q}{m}E_z - \frac{q^2}{2m^2} r\left(\frac{r_0^2}{r^2}B_{0z} - B_z\right)B_r\right]\cdot r' = \frac{q}{m}\frac{I}{2\pi\varepsilon_0 v_z r} + \left(\frac{q}{2m}\right)^2 \frac{r_0^4 B_{0z}^2}{r^3} - \left(\frac{q}{2m}\right)^2 B_z^2 r \quad (9.7b)$$

Now we investigate special cases of the paraxial equation. If we assume no acceleration (E_z=0) and very small B_r << B_z and B_{0z}=0 (at the source) we get

$$v_z^2 r'' + \left(\frac{q}{2m}\right)^2 B_z^2 r - \frac{q}{m}E_r = 0$$

The space charge term can be expressed as $E_r = \frac{\rho_0}{2\varepsilon_0} r$

$$v_z^2 r'' + \left[\left(\frac{q}{2m}\right)^2 B_z^2 - \frac{q}{2m}\frac{\rho_0}{\varepsilon_0}\right] r = 0$$

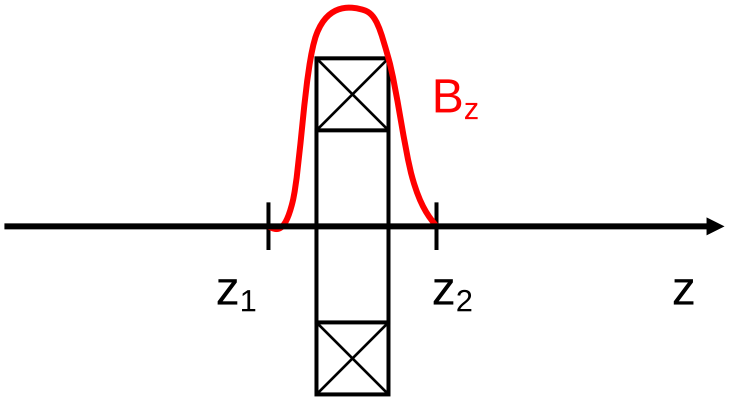
So even with space charge we get $r'' + K\cdot r = 0$ Hillsche DGL

This equation will be the basis of the linear beam optics. The space charge term we will explain in another lecture.

Beam focusing with axial symmetric fields without space charge:

1) Magnetic solenoid lens

$B_{0z} = E_z = E_r = 0$



$$v_z^2 r'' + \left[\frac{q^2}{2m^2} r B_z B_r \right] \cdot r' + \left(\frac{q}{2m} \right)^2 B_z^2 r = 0$$

In the integral from z_1 to z_2 the effects of B_r will cancel each other out to a large extend, because the field is symmetric around z. Now one can solve the equation above numerically with space charge, or without space charge analytically:

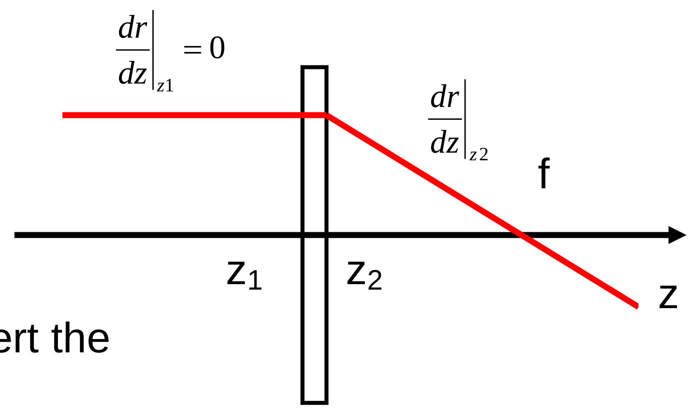
$$v_z^2 r'' + \left(\frac{q}{2m} \right)^2 B_z^2 r = 0 \quad \Leftrightarrow \quad r'' + k \cdot r = 0$$

Hill`s differential equation (9.8)

$$k = \left(\frac{q}{2m} \right)^2 \frac{B_z^2}{v_z^2} = \frac{q^2}{4} \frac{B_z^2}{p_z^2} = \frac{B_z^2}{4(B\rho)^2}$$

The focusing of a solenoid is chromatic (v_z or p_z-dependent).

$$\left.\frac{dr}{dz}\right|_{z1}^{z2} = -\left(\frac{q}{2m}\right)^2 \frac{1}{v_z^2} \int_{z1}^{z2} B_z^2 r \, dz \approx -\frac{q}{8m} \frac{r_0}{U_{acc}} \int_{z1}^{z2} B_z^2 \, dz$$



Here a thin lens has been assumed. Therefore, the beam radius r_0 within the lens is nearly constant. For the velocity one can insert the acceleration voltage ***U_{acc}.***

Therewith the focal length is:
$$\left.\frac{dr}{dz}\right|_{z2} \approx -\frac{r_0}{f} \quad \Rightarrow \quad \frac{1}{f} = \frac{q}{8m \cdot U_{acc}} \int_{z1}^{z2} B_z^2 \, dz \qquad (9.9)$$

In linear beam optics an effective field length z_L is used with $B_0^2 z_L = \int_{z1}^{z2} B_z^2 \, dz$.

Charged particles gyrate within the solenoid field. Due to the Busch theorem $\dot{\theta} = -\frac{q}{2m} B_z$.

with $\dot{\theta} = \frac{d\theta}{dt} = \frac{d\theta}{dz} v_z$ it follows for the change in angle with respect to the position:

$$\frac{d\theta}{dz} = -\frac{q}{2m \cdot v_z} B_z = -\frac{q}{\sqrt{8mqU_{acc}}} B_z$$

$$\Rightarrow \quad \theta = -\frac{q}{\sqrt{8mqU_{acc}}} \int_{z1}^{z2} B_z dz \qquad (9.10)$$

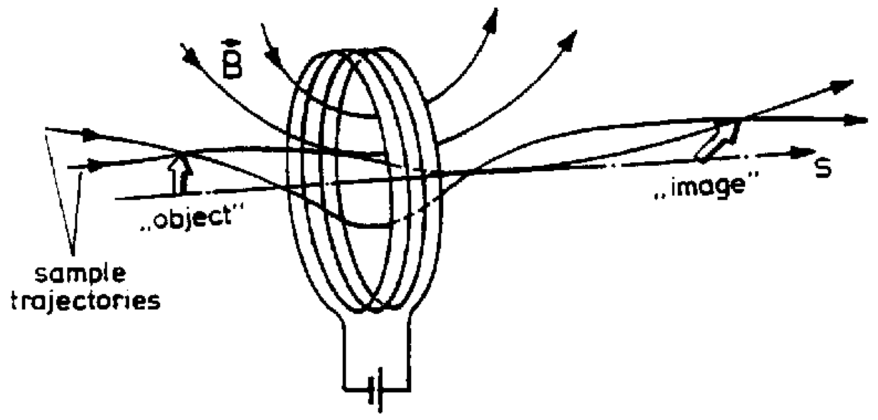


Figure 8: Particle trajectories and field lines in a "thin" lens formed by the solenoidal field of a coil (according to Bergmann/Schäfer: Optik)

Beside the radial forces there are also azimuthal forces, which lead to a change in angle in this direction.

The change in angle depends on the sign of the charge and the direction of the magnetic field. Advantages of a solenoid lens are the preservation of space charge compensation and the absence of spherical aberration. Disadvantages are the limitation to low energy beams and the chromatic aberration.

Electrostatic cylindrical lenses

$B_z = B_r = 0$,

$$v_z^2 r'' + \left[\frac{q}{m} E_z\right] \cdot r' = \frac{q}{m} E_r \qquad (9.11a)$$

The aim is to describe the fields with there potentials:

$$v_z^2 = \frac{2qU_{acc}}{m}$$

$$r'' + \frac{E_z}{2U_{acc}} \cdot r' - \frac{E_r}{2U_{acc}} = 0 \qquad (9.11b)$$

$$E_z = -\frac{\partial V}{\partial z}, \quad E_r = -\frac{\partial V}{\partial r}, \quad E_\theta = 0 \quad and \quad \Delta V = \frac{\partial^2 V}{\partial z^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2} = 0$$

For axial symmetric problems the potential can be expanded:

$$V(z,r) = V(z,0) + \frac{1}{2} r^2 \frac{\partial^2 V(z,0)}{\partial r^2} + \frac{1}{24} r^4 \frac{\partial^4 V(z,0)}{\partial r^4} + \ldots$$

with the Laplace-equation one gets

$$V(z,r) \approx V(z,0) - \frac{1}{2} r^2 \frac{\partial^2 V(z,0)}{\partial z^2} - \frac{1}{2} r \frac{\partial V(z,0)}{\partial r}$$

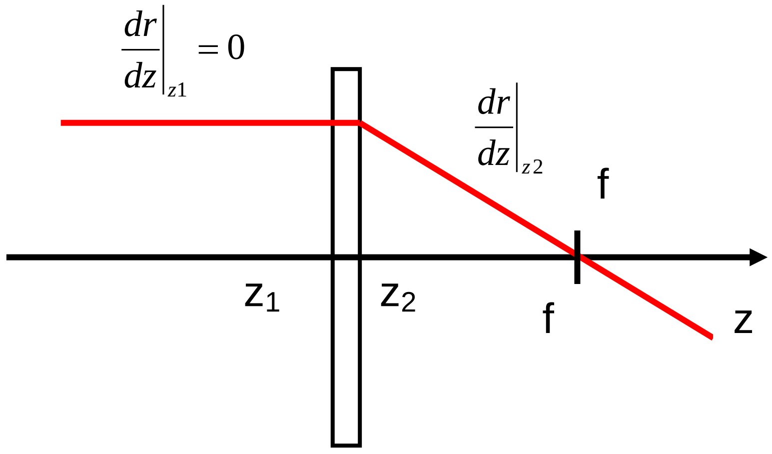
Additionally it is: $r\frac{\partial V}{\partial r} = -\frac{1}{2} r^2 \frac{\partial^2 V}{\partial z^2} \Rightarrow \quad V(z,r) \approx V(z,0) - \frac{1}{4} r^2 \frac{\partial^2 V(z,0)}{\partial z^2}$

Therewith the electric field strength is

$$E_z = -\frac{\partial V}{\partial z} = -V', \quad E_r = -\frac{\partial V}{\partial r} = \frac{1}{2} r \frac{\partial^2 V}{\partial z^2} = \frac{1}{2} r V''$$

The equation of motion within electrostatic cylindrical lenses is given by

$$r'' + \frac{V'}{2U_{acc}} \cdot r' - \frac{V''}{4U_{acc}} r = 0 \quad mit \quad U_{acc} = U_0 - V \qquad (9.12)$$



$$\frac{d}{dz}\left(\sqrt{U_0 - V}\,\frac{dr}{dz}\right) = \frac{r}{4}\frac{V''}{\sqrt{U_0 - V}} \quad \Rightarrow$$

$$\left.\sqrt{U_0 - V}\,\frac{dr}{dz}\right|_{z1}^{z2} = \left.\sqrt{U_0 - V_{z2}}\,\frac{dr}{dz}\right|_{z2} = \int_{z1}^{z2} \frac{r}{4}\frac{V''}{\sqrt{U_0 - V}}\,dz$$

$$\frac{1}{f} = -\frac{1}{r_0\sqrt{U_0 - V_{z2}}} \int_{z1}^{z2} \frac{r}{4}\frac{V''}{\sqrt{U_0 - V}}\,dz \qquad (9.13)$$

$$\frac{1}{f} = -\frac{1}{r_0}\left.\frac{dr}{dz}\right|_{z2}$$

The integral can be written as:

$$\int_{z1}^{z2} \frac{r}{4} \frac{V''}{\sqrt{U_0 - V}} dz \approx \frac{r_0}{4} \int_{z1}^{z2} \frac{V''}{\sqrt{U_0 - V}} dz = \frac{r_0}{4} \left[\frac{V'}{\sqrt{U_0 - V}} \right]_{z1}^{z2} - \frac{r_0}{8} \int_{z1}^{z2} \frac{(V')^2}{(U_0 - V)^{3/2}} dz$$

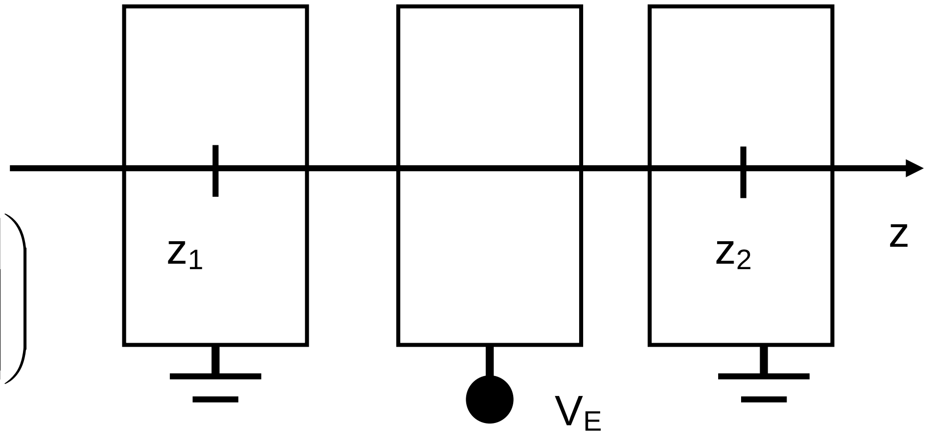
$$\frac{1}{f} = -\frac{1}{r_0 \sqrt{U_0 - V_{z2}}} \int_{z1}^{z2} \frac{r}{4} \frac{V''}{\sqrt{U_0 - V}} dz = \frac{1}{4\sqrt{U_0 - V_{z2}}} \left(\frac{1}{2} \int_{z1}^{z2} \frac{(V')^2}{(U_0 + V)^{3/2}} dz - \left[\frac{V'}{\sqrt{U_0 - V}} \right]_{z1}^{z2} \right) \tag{9.13b}$$

Special cases:

A) The Einzel lens

$V_{z1} = V_{z2} = 0$

$$\frac{1}{f} = \frac{1}{4\sqrt{U_0}} \left(\frac{1}{2} \int_{z1}^{z2} \frac{(V')^2}{(U_0 - V)^{3/2}} dz - \left[\frac{V'_{z2}}{\sqrt{U_0}} - \frac{V'_{z1}}{\sqrt{U_0}} \right] \right)$$



If the field strengths are $V'_{z1} = V'_{z2} = 0$, then the focal length is

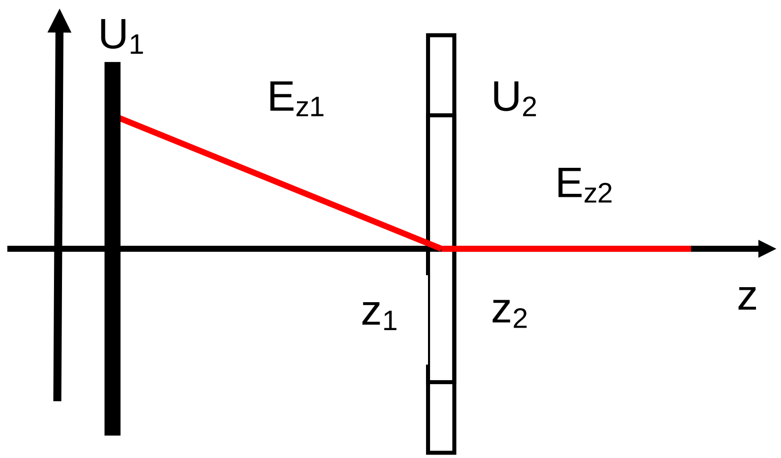
$$\frac{1}{f} = \frac{1}{8\sqrt{U_0}} \int_{z1}^{z2} \frac{(V')^2}{(U_0 - V)^{3/2}} dz \qquad (9.14)$$

That means that all single lenses are focusing lenses, because all parts in the equation are positive. For beams with high space charge the lens has the disadvantage of breaking the space charge compensation.

B) Diaphragm

From (9.13b) we get

$$\frac{1}{f} = \frac{1}{4\sqrt{U_0 - V_{z2}}} \left(- \left[\frac{V'_{z2}}{\sqrt{U_0 - V_{z2}}} - \frac{V'_{z1}}{\sqrt{U_0 - V_{z1}}} \right] \right)$$



It is assumed that $z_1 \rightarrow z_2$ and therefore the integral vanishes. Now we have $V_{z1} = V_{z2} = 0$.

$$\frac{1}{f} \approx \frac{1}{4\sqrt{U_0}} \frac{V'_{z2} - V'_{z1}}{\sqrt{U_0}} \approx \frac{1}{4U_0}(E_{z2} - E_{z1}) \qquad (9.15)$$

The focal length is thereby proportional to the difference of the field strengths in front and behind the aperture. If $V'_{z1} = E_1 \geq V'_{z2} = E_2$, which is the case if E_2=0, then f is negative. Therefore, an anode aperture is a divergent lens for all extraction systems. This effect is called the **anode lens effect**.

$$f \approx -\frac{4U_0}{E_1} = -\frac{4U_0}{E_{Anode}} \tag{9.16}$$

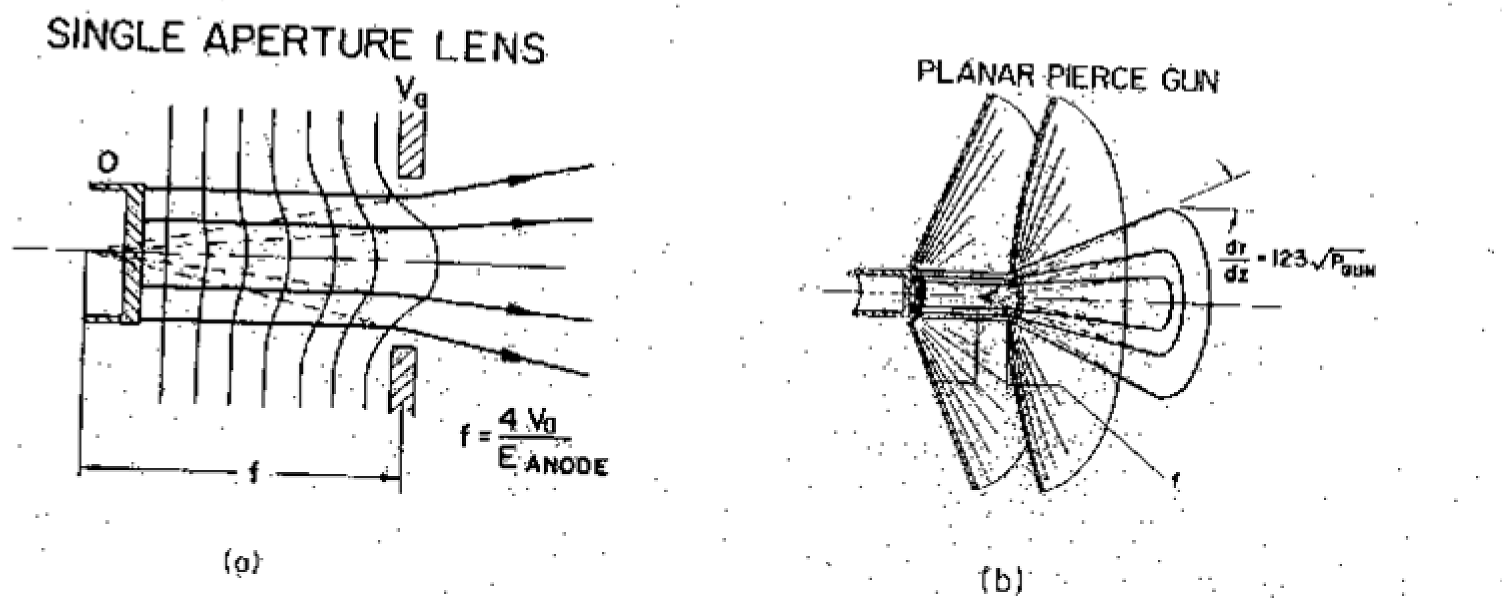


FIG. 8. Illustrations of (a) the lens effect of the transverse fields in the anode aperture of (b) a planar Pierce gun.

This defocusing effect can be calculated for a diode set-up:

$$f = -\frac{4U_0}{E_{Anode}} = -\frac{4U_{Anode}}{E_{Anode}}$$

We have from Child-Langmuir

$$U_{Anode} = A \cdot d^{4/3} = \left(\frac{9}{4} \frac{j}{\varepsilon_0 \sqrt{\frac{2e}{m}}} \right) d^{4/3} \qquad \Rightarrow \qquad E_{Anode} = \left. \frac{dU_{Anode}}{dx} \right|_d = \frac{4}{3} A \cdot d^{1/3}$$

Therefore we get:

$$f = -\frac{4U_{Anode}}{E_{Anode}} = -\frac{4Ad^{4/3}}{\frac{4}{3} Ad^{1/3}} = -3d$$

The anode lens effect acts as a defocusing lens with the focusing length of $f = -3d$, three times the electrode distance of a diode.