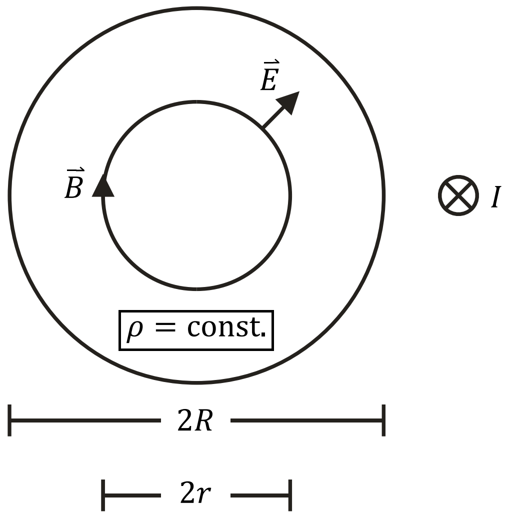
Space Charge

- space charge forces
- tune shift
- emittance growth
- beam matching

$$\vec{\nabla}\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\int \vec{\nabla}\vec{E}\,dV = \int \vec{E}\,dA$$

$$\Longrightarrow E(r) = \frac{\rho \cdot r}{2\varepsilon_0}$$



$$I = \frac{\partial Q}{\partial t} = \frac{\rho\pi R^2 \cdot \partial L}{\partial t}$$

$$= \rho\pi R^2 \cdot \beta \cdot c$$

$$\boxed{E(r) = \frac{I \cdot r}{2\pi\varepsilon_0 \cdot R^2 \cdot \beta \cdot c}}$$ repulsive!

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\boxed{\int (\vec{\nabla} \times \vec{B})\,d\vec{A} = \oint \vec{B}\,d\vec{s}}$$ attractive!

$$B(r) = \frac{\mu_0 \cdot I \cdot r}{2\pi R^2}$$

$$\ddot{\vec{r}}_{sc} = \frac{e \cdot q}{A \cdot m_0 \cdot \gamma}\left[\vec{E} - \vec{v} \times \vec{B}\right] = \boxed{\ddot{r}_{sc} = \frac{e \cdot q \cdot I}{A \cdot m_0 \cdot \beta \cdot \gamma^3 \cdot 2\pi\varepsilon_0 R^2 \cdot c} \cdot r}$$ linear in r!

- $\ddot{r}_{sc} \sim r \rightarrow$ sc acts like defocusing quad with $k(R)$
- $\ddot{r}_{sc}$ decreases with energy, $\beta = 1 \rightarrow \ddot{r} = 0$

sc adds to Hill's equations:

$r \rightarrow x, \qquad x'' = \frac{1}{\beta^2 c^2}\ddot{x}$

$\rightarrow x'' + \left[k(s) - \frac{P}{R^2(s)}\right] x = 0,$

"Perveance" $P = \frac{e \cdot q \cdot I}{2\pi\varepsilon_0 \cdot A \cdot m_0 \cdot c^3 \beta^3 \gamma^3}$
dimensionless

$x'' + k_{eff}(s) = 0$

still linear equation $\rightarrow \varepsilon_{rms}^2 = det\begin{vmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle x'x \rangle & \langle x'^2 \rangle \end{vmatrix}$ preserved!

assumption:
- $k(s) = \text{const.} > 0$
- $R(s) = \text{const.}$

$\Big\}\ x'' \neq 0!$

Space Charge

→ $x(s)$ like harmonic oscillator $\sim e^{i\sqrt{k_{eff}}\cdot s}$

$\sigma_{eff} \coloneqq \frac{\text{phase advance}}{\text{length}} = \sqrt{k_{eff}}$, $\quad I \neq 0$

$\sigma_o \coloneqq \sqrt{k}$, $\quad I = 0$

→ $\boxed{\sigma_{eff}^2 = \sigma_0^2 - \frac{P}{R^2} \coloneqq \sigma_0^2 - \text{sc-tune shift}}$

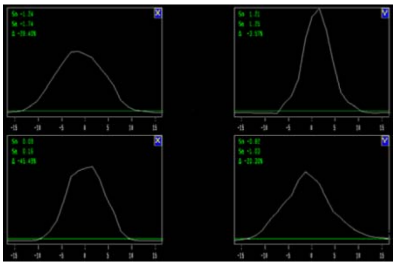
$\sigma_{eff}^2 > 0$ limits achievable current

$\rho =$ const. generally not true, generally ρ decreases with r

$$\rightarrow E(r) = \frac{\int_0^r \rho(r')r'dr'}{\varepsilon_0 \cdot r} \rightarrow x'' + [k(s) + f(x)] \cdot x = 0 \quad \text{non-linear}$$

- each particle sees different k_{eff}
- ε_{rms} not preserved
- ε_{rms} will grow

→ sc limits currents in accelerators

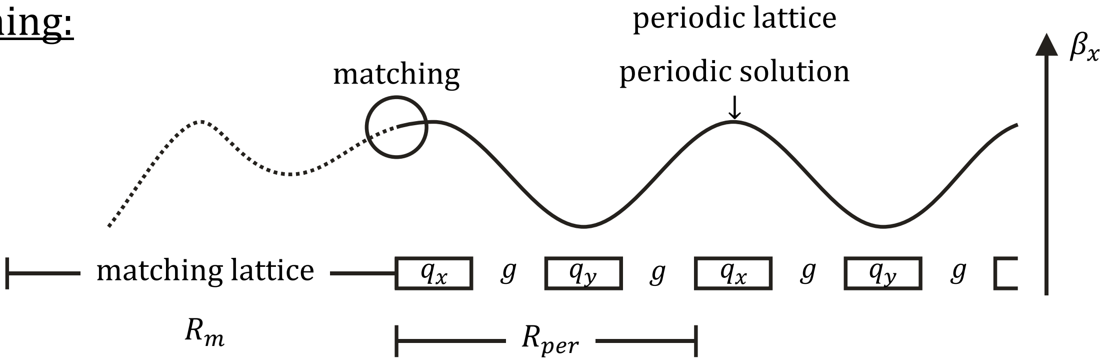


Beam Matching:

beam moments:

$\beta_b = \frac{\langle xx \rangle}{\varepsilon_{rms}}$

$\alpha_b = -\frac{\langle xx' \rangle}{\varepsilon_{rms}}$



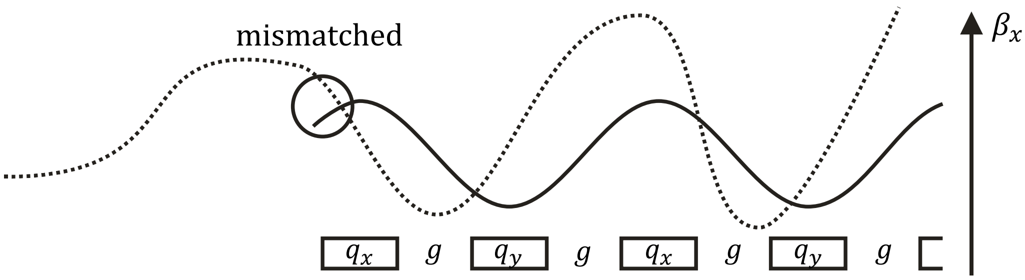
definition of periodic solution: $\begin{bmatrix} \beta_{per} & -\alpha_{per} \\ -\alpha_{per} & \beta_{per} \end{bmatrix} = R_{per} \cdot \begin{bmatrix} \beta_{per} & -\alpha_{per} \\ -\alpha_{per} & \beta_{per} \end{bmatrix} \cdot R_{per}^{\mathrm{T}}$

- depend just from R_{per}
- independent from beam

$\boxed{\text{matched transport in periodic lattice: } \beta_b = \beta_{per}, \;\; \alpha_b = \alpha_{per}}$

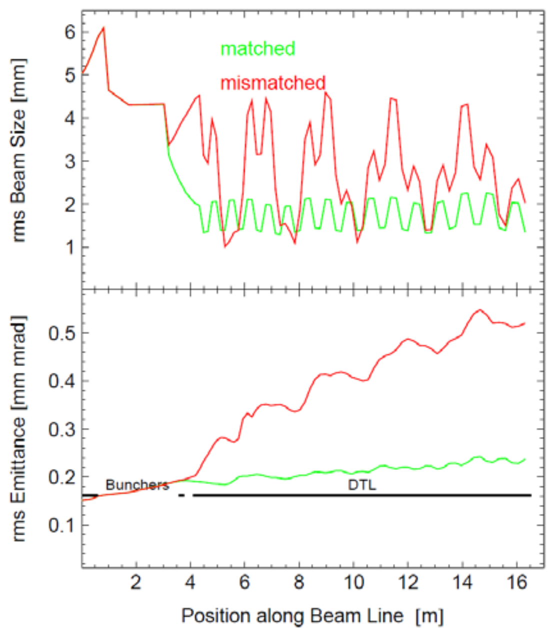
<u>"matching"</u>: set R_m such that:

- β_b @ end of matching lattice $= \beta_{per}$ @ start of periodic lattice
- α_b @ end of matching lattice $= \alpha_{per}$ @ start of periodic lattice



matched transport: minimizes $\Delta\varepsilon_{rms}$ along periodic lattices for beams with $\rho = \rho(r)$

mismatch ≙ free energy transferred to $\Delta\varepsilon_{rms}$



Matching in practice

① > 4 quads, > 2 buncher } > 6 parameters to match 6 quantities $\beta_x, \alpha_x, \cdots, \beta_z, \alpha_z$

matching lattice R_m | periodic lattice R_{per}

- @ ① beam diagnostics to measure $\varepsilon_{rms}, \beta_b, \alpha_b$ in x, y, z
- choose quad- & buncher settings for matched transport
- numerical tools for that:
  - $I = 0$:
    - solve envelope equation ($I = 0$) piecewise
    - initial conditions from ①
    - final result must be periodic solution
    - codes like MAD, MIRKO, TRANSPORT

Space Charge

- $I \neq 0$:
  - beam with $\rho(r) \longrightarrow$ 6D-ellipsoid with $\rho = \text{const.}$
  - beam-dimensions from
    - $\langle xx \rangle_{ell} \equiv \langle xx \rangle_{org}$
    - $\langle xx' \rangle_{ell} \equiv \langle xx' \rangle_{org}$

      $\vdots$
    - $\langle z'z' \rangle_{ell} \equiv \langle z'z' \rangle_{org}$
  - homogeneous ellipsoids have $\Delta\varepsilon_{rms} = 0$
  - homogeneous ellipsoids can be tracked numerically

    → code TRACE-3D
  - input from ①
  - final result must be periodic solution