Transformation of phase space in a circular accelerator

In a circular accelerator the coefficients k(s) of Hill's equation must be periodic:

$$r'' + k(s)\cdot r = 0 \qquad k(s+C) = k(s)$$

C is the circumference of the accelerator. A solution of this equation is derived by the Floquet-theorem with

$$r = \varphi(s) \quad \Rightarrow \quad \varphi(s+C) = e^{i\Delta\psi}\varphi(s)$$

$\Delta\psi$ is the phase advance after one period. One solution was derived:

$$x(s) = \sqrt{\varepsilon\beta(s)}\cdot\cos\left(\Psi(s)+\Psi_0\right)$$

and for the transformation matrix

$$\underline{A} = \begin{pmatrix} \sqrt{\frac{\beta_f}{\beta_i}}(\cos\Delta\Psi + \alpha_i\sin\Delta\Psi) & \sqrt{\beta_f\cdot\beta_i}\sin\Delta\Psi \\ \frac{(\alpha_i-\alpha_f)\cos\Delta\Psi - (1+\alpha_i\alpha_f)\sin\Delta\Psi}{\sqrt{\beta_f\cdot\beta_i}} & \sqrt{\frac{\beta_i}{\beta_f}}(\cos\Delta\Psi - \alpha\sin\Delta\Psi) \end{pmatrix}$$

Due to the periodicity for the Twiss parameter we get $\beta(s+C) = \beta(s)$ and $\alpha(s+C) = \alpha(s)$. For the eigen-solution of the transfer matrix of a circular accelerator we get:

$$\underline{R}_{s\to s+C} = \underline{M} = \begin{pmatrix} \cos\Delta\Psi + \alpha(s)\sin\Delta\Psi & \beta(s)\sin\Delta\Psi \\ -\frac{(1+\alpha^2(s))\sin\Delta\Psi}{\beta(s)} & \cos\Delta\Psi - \alpha(s)\sin\Delta\Psi) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\Delta\Psi + \alpha(s)\sin\Delta\Psi & \beta(s)\sin\Delta\Psi \\ -\gamma(s)\sin\Delta\Psi & \cos\Delta\Psi - \alpha(s)\sin\Delta\Psi) \end{pmatrix}$$

This matrix does the transformation of x and x' for one revolution in the ring.

$$\begin{pmatrix} x(s+C) \\ x'(s+C) \end{pmatrix} = \begin{pmatrix} c(s,s+C) & s(s,s+C) \\ c'(s,s+C) & s'(s,s+C) \end{pmatrix}\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \underline{R}(s,s+C)\,\vec{x}$$

Dispersion and resonances

$$\underline{M}(s) = \begin{pmatrix} \cos\Delta\Psi + \alpha(s)\sin\Delta\Psi & \beta(s)\sin\Delta\Psi \\ \frac{(1+\alpha^2(s))\sin\Delta\Psi}{\beta(s)} & \cos\Delta\Psi - \alpha(s)\sin\Delta\Psi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\mu + \alpha(s)\sin\mu & \beta(s)\sin\mu \\ -\gamma(s)\sin\mu & \cos\mu - \alpha(s)\sin\mu \end{pmatrix} = \cos\mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin\mu \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix}$$

$$= \cos\mu\, \underline{I} + \sin\mu\, \underline{J}$$

Twiss matrix: **M**

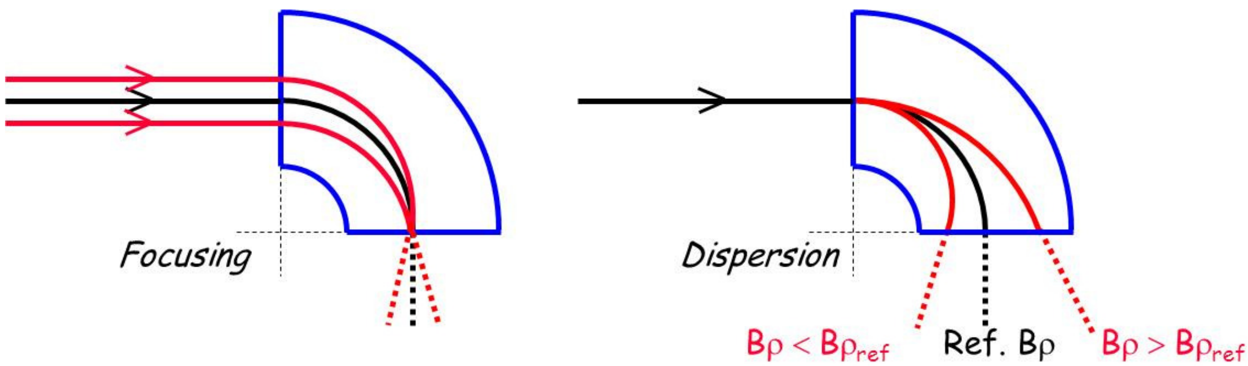
$$\det\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = -\alpha^2 + \beta\cdot\gamma = 1 \qquad , \underline{J}\cdot\underline{J} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\underline{I} \quad , \underline{J}^{-1} = -\underline{J}$$

For n revolutions: $\underline{M}^N = (\cos\mu\, \underline{I} + \sin\mu\, \underline{J})^N = \cos N\mu\, \underline{I} + \sin N\mu\, \underline{J}$

The beam matrix $\underline{S} = \varepsilon\begin{pmatrix} \beta & -\alpha \\ \alpha & \gamma \end{pmatrix}$ is transformed via M to $\underline{S} = \underline{M}\cdot\underline{S}\cdot\underline{M}^T$ as required by the periodicity.

Dispersion

As the magnetic forces of the magnets in an accelerator depend on the momentum of the particle, the bending and focusing of the charged particles depends on the particle momentum.



Dispersion and resonances

Particles entering with a different energy (i.e. different Bρ) are bend with a different radius. Therefore a bending magnet does translate a momentum or energy difference to the perfect particle into a spacial offset of the particle after the bend. This couples the longitudinal with the transverse phase space. Therefore we extend the matrix formalism to 6 dimensions:

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ \lambda \\ \delta \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & \cdots & R_{16} \\ R_{21} & R_{22} & \cdots & R_{26} \\ \vdots & \vdots & \ddots & \vdots \\ R_{61} & R_{62} & \ldots & R_{66} \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \\ \lambda_0 \\ \delta_0 \end{pmatrix}$$

x = offset in x-direction [m]

x' = slope in x-direction [rad]

y = offset in y-direction [m]

y' = slope in y-direction [rad]

λ = longitudinal offset from synchronous particle [m]

$\delta = \Delta p/p_0$ = relative momentum difference

It is usually convenient to look at the matrix using 2x2 block matrices

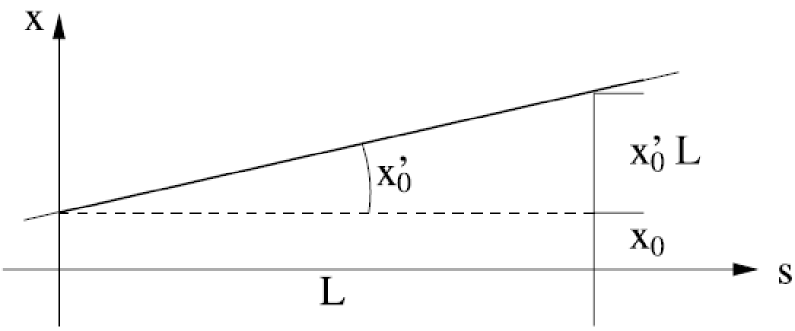
$$R = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix}$$

and in the absence of coupling

$$R = \begin{pmatrix} R_{xx} & 0 & 0 \\ 0 & R_{yy} & 0 \\ 0 & 0 & R_{zz} \end{pmatrix}$$

Example: The drift in the 6-dimensional transfer matrix

$$\frac{\Delta v}{v_0} = \frac{1}{\gamma^2}\frac{\Delta p}{p_0} = \frac{1}{\gamma^2}\cdot\delta_0 \quad ==> \quad \lambda(s) - \lambda_0 = L\frac{\Delta v}{v_0} = \frac{L}{\gamma^2}\cdot\delta_0 \quad \Rightarrow \lambda(s) = \lambda_0 + \frac{L}{\gamma^2}\cdot\delta_0$$



$$x = x_0 + L\,x_0'$$
$$x' = x_0'$$
$$y = y_0 + L\,y_0'$$
$$y' = y_0'$$
$$\lambda = \lambda_0 + \left(L/\gamma^2\right)\delta_0$$
$$\delta = \delta_0$$

Dispersion and resonances

$$R_{Drift} = \begin{pmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L/\gamma^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

More general for a bending magnet for instance:

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ \lambda \\ \delta \end{pmatrix} = \begin{pmatrix} C_x(s) & S_x(s) & 0 & 0 & 0 & D_x(s) \\ C_x'(s) & S_x'(s) & 0 & 0 & 0 & D_x'(s) \\ 0 & 0 & C_y(s) & S_y(s) & 0 & 0 \\ 0 & 0 & C_y'(s) & S_y'(s) & 0 & 0 \\ (\lambda \,|\, x_0) & (\lambda \,|\, x'_0) & 0 & 0 & (\lambda \,|\, \lambda_0) & (\lambda \,|\, \delta_0) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \\ \lambda_0 \\ \delta_0 \end{pmatrix}$$

The dispersion D(s) couples the momentum delta to a transverse offset: $x_D = D_x(s) \cdot \delta_0$

In case of a bending magnet we talk about dispersion, in case of a magnetic lens about **chromatic aberration.** We get the according expressions for the dispersion by the equation of motion.

The equation of motion that describes the dispersive case (transverse beam dynamics reminder):

$$x''(s) + \left(\frac{1}{\rho^2(s)} + k(s) \right) x(s) = \frac{1}{\rho(s)} \frac{\Delta p}{p_0}$$

$$y''(s) - k(s)\; y(s) = 0$$

Both equations can be transformed in Hill's-equations:

$$k_x(s) = k(s) + \frac{1}{\rho^2(s)} \quad ; \quad k_y(s) = -k(s)$$

Now we take the momentum spread into account:

Dispersion and resonances

$$x'' + k_x(s)\cdot x = h(s)\cdot\delta \qquad with \qquad h(s) = \frac{1}{\rho_0(s)} \quad ; \quad \delta = \frac{\Delta p}{p_0}$$

In order to solve the equation, one needs to find the modified orbit $x_D(s) = D(s)\cdot\delta$ caused by the momentum difference. This is a periodic function and is called dispersion. The solution is $x_\delta(s) = x_D(s) + x(s)$.where x(s) is the betatron oscillation and the solution of the homogenous Hill's equation.

$$D''(s) + k_x(s)D(s) = h(s)$$

For the dispersion function we have D(s) = D(s+C) and D'(s) = D'(s+C). Boundary conditions are $s_0=0$, $D(s_0)=D_0$ and $D'(s_0)=D'_0$.

$$D(s) = D_0 C(s) + D_0' S(s) \qquad \begin{matrix} C(s_0)=1 & ; & S(s_0)=0 \\ C'(s_0)=0 & ; & S'(s_0)=1 \end{matrix}$$

With the cosine and sine like functions C(s) and S(s). As a solution for D(s) we get (see Hinterberger)

$$D(s) = \frac{\sqrt{\beta(s)}}{2\sin\frac{\mu}{2}}\cdot\int_s^{s+C} h(\bar{s})\sqrt{\beta(\bar{s})}\cos(\Delta\Psi - \frac{\mu}{2})\,d\bar{s}$$

Here ΔΨ is the phase advance dependent on s and μ the phase advance per revolution period.

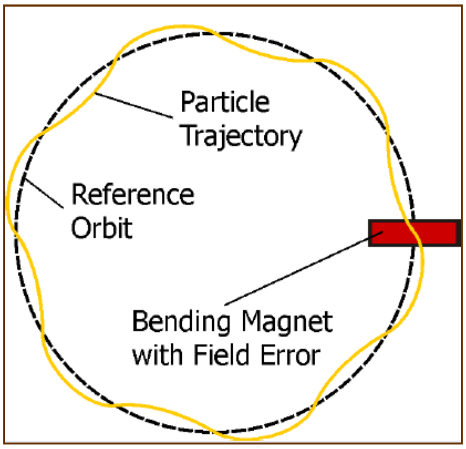
D(s) is increasing with $\sin\frac{\mu}{2} \to 0$. In case of $\sin\frac{\mu}{2} = 0 \quad \Rightarrow \quad \mu = 2\pi\cdot N$ diverges.

Now μ determines the number of betatron oscillations per revolution period

$Q = \frac{\mu}{2\pi}$ which is the so called **"betatron tune"**.

Now we remember that

$$\Psi(s) = \int\frac{ds}{\beta(s)} \quad \Rightarrow \quad Q = \frac{\mu}{2\pi} = \frac{1}{2\pi}\int_0^C\frac{ds}{\beta(s)}$$



Q_x, and Q_y are the **Q-values or working points** of the circular accelerator.

Momentum compaction and Resonances

In a circular accelerator we have ω = ω(p), da = 2π*v/C. Damit erhält man

$$\frac{\Delta\omega}{\omega_0} = \frac{\Delta v}{v_0} - \frac{\Delta C}{C_0} \quad \text{with} \qquad \frac{\Delta v}{v_0} = \frac{1}{\gamma^2}\frac{\Delta p}{p_0} \qquad \frac{\Delta C}{C_0} = \alpha_p \frac{\Delta p}{p_0}$$

α_p is called the Momentum Compaction Factor. This parameter is the connection between the relative difference of the orbit length per revolution and the relative momentum difference of a particle.

$$\frac{\Delta\omega}{\omega_0} = \left(\frac{1}{\gamma^2} - \alpha_p\right)\frac{\Delta p}{p_0} = \eta\frac{\Delta p}{p_0}$$

As a good approximation $x_D(s) = D(s)\cdot\delta$ does only contribute in the bending magnets to $\Delta C/C_0$. Therefore the length difference of different orbits is (with $d\alpha = ds/\rho_0 = h(s)ds$)

$$\Delta C = \int_s^{s+C_0} (\rho_0 + x_D)d\alpha - \int_s^{s+C_0} \rho_0 d\alpha = \int_s^{s+C_0} h\cdot x_D d\bar{s} = \frac{\Delta p}{p_0}\int_s^{s+C_0} D(\bar{s})h(\bar{s})d\bar{s}$$

$$\rightarrow \quad \alpha_p = \frac{1}{C_0}\int_s^{s+C_0} D(\bar{s})h(\bar{s})d\bar{s}$$

We can vary D(s) via the ion optics and therewith Δp as well. α_p determines the dispersion in the area of the bending magnets. The smaller D(s) the closer the orbits for a given $\Delta p/p_0$ and the smaller is α_p.

$$\eta = \left(\frac{1}{\gamma_{tr}^2} - \alpha_p\right) = 0 \rightarrow E_{tr} = \gamma_{tr} m_0 c^2 \rightarrow \eta = \frac{1}{\gamma^2} - \frac{1}{\gamma_{tr}^2}$$

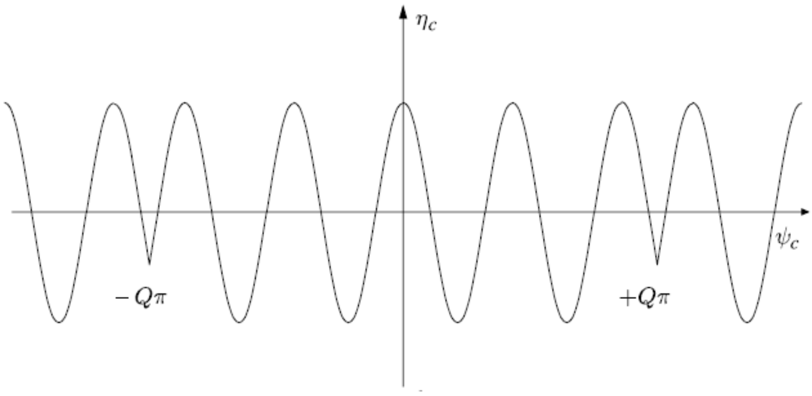
Dispersion and resonances

$$\gamma < \gamma_{tr} \quad \Leftrightarrow \quad \eta > 0 \ ,$$
$$\gamma = \gamma_{tr} \quad \Leftrightarrow \quad \eta = 0$$
$$\gamma > \gamma_{tr} \quad \Leftrightarrow \quad \eta < 0$$

At $\gamma = \gamma_{tr} \quad \Leftrightarrow \quad \frac{\Delta\omega}{\omega_0} = 0$ the particles orbit isochronous in the ring, independent on the momentum (and energy). This transition is special for heavy ion synchrotrons and storage rings. For a strong focusing synchrotron we get

$$\alpha_p \approx \frac{1}{Q_x^2}, \quad \gamma_{tr} \approx Q_x$$

An error of the dipole field δB at s= s_0, which act over a very short length of Δs establishes a kick of the beam (change in the angle) Δx'. The „closed orbit“, which is the orbit of the perfect beam particle, is distorted → ”closed orbit distortion“.



Such an distortion causes betatron oscillations.

$$\Delta x' = \frac{-\delta B}{B\rho}\Delta s = F(s_0)\Delta s \ \rightarrow \ x'' + k_x(s)x = F(s)$$

solution: $$x(s) = \frac{\sqrt{\beta(s)}}{2\sin Q\pi} \cdot \int_s^{s+C_0} F(\bar{s})\sqrt{\beta(\bar{s})}\cos(\Delta\Psi - Q\pi)\, d\bar{s}$$

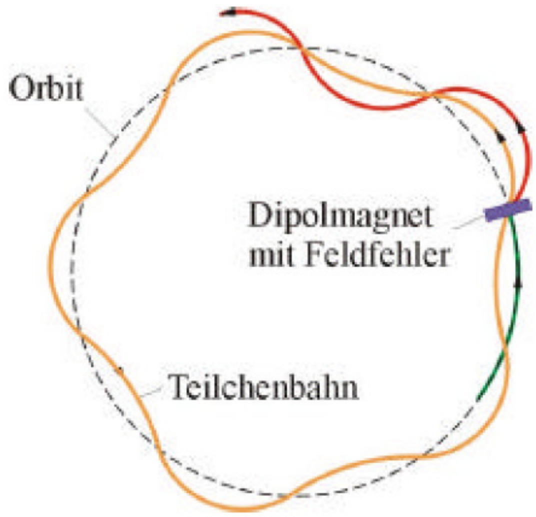
$$= \frac{\sqrt{\beta(s)}}{2\sin Q\pi} \cdot \Delta x' \sqrt{\beta(s_0)}\cos(\Delta\Psi - Q\pi) \quad \Rightarrow \quad \frac{x(s)}{\sqrt{\beta(s)}} = \frac{\sqrt{\beta(s_0)} \cdot \Delta x'}{2\sin Q\pi}\cos(\Delta\Psi - Q\pi)$$

Dispersion and resonances

Amplitude: $a = \dfrac{\sqrt{\beta(s_0)} \cdot \Delta x'}{2 \sin Q\pi}$. The amplitude is proportional to the kick strength, to the betatron function to 1/ sin(Qπ). The amplitude diverges with Q=N and the particles get lost.

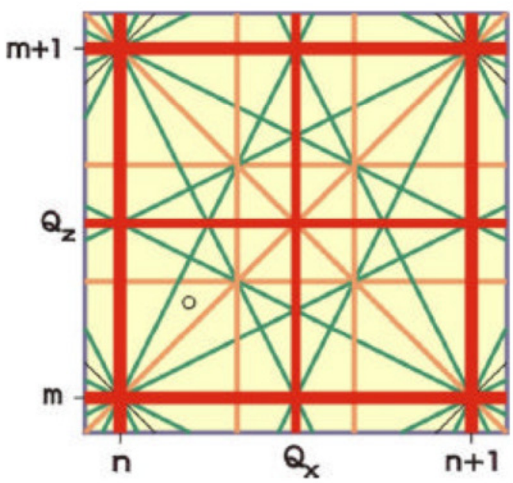
➔ stop band

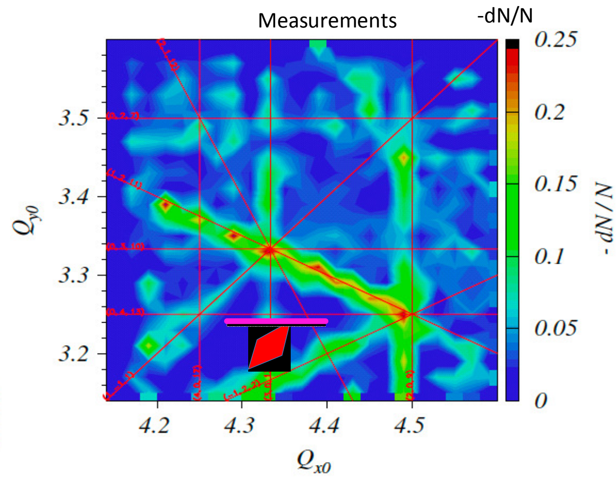
The particles pass the error region with the same phase and errors add to increasing amplitudes.
Closed orbit distortions are compensated with steerer magnets.



Feldfehler	optische Resonanz
Dipolfehler	$Q = n$
Quadrupolfehler	$Q = n + 1/2$
Sextupolfehler	$Q = n + 1/3$
Oktupolfehler	$Q = n + 1/4$
usw.	usw.

In a [illegible] ders are sketched:





G. Franchetti et al., GSI-Acc-Note-2005-02-001

$$m\,Q_x + n\,Q_z = p \qquad (m, n, p = \text{ganze Zahlen})$$