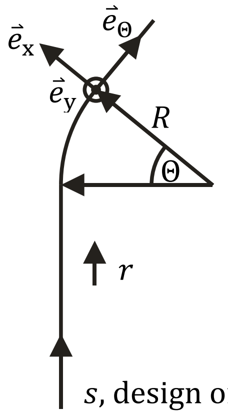
Transverse Dynamics

- transverse differential equations
- envelope equation, phase space ellipse
- matrix formalism
- Twiss parameters to quantify beam
- focusing lattices

Transverse Differential Equations



$$\begin{bmatrix} x \\ y \\ \Theta \end{bmatrix}$$

$$\dot{\vec{e}}_x = \dot{\Theta}\vec{e}_\Theta$$

$$\dot{\vec{e}}_y = 0$$

$$\dot{\vec{e}}_\Theta = -\dot{\Theta}\vec{e}_x$$

s, design orbit: from dipoles $B_{y,d} \cdot R = \frac{p_0}{q \cdot e}$

- off. orbit part.: $B_{y,q} = G_x \cdot x, \quad B_{x,q} = G_y \cdot y$ $\qquad \vec{\nabla} \times \vec{B} = 0 \rightarrow G_x = G_y$
- linear corrections in x & y
- design part (x=y=0!) → no corrections

part. position (transverse): $\vec{r} = (R + x)\vec{e}_x + y \cdot \vec{e}_y$

$$\dot{\vec{r}} = \dot{x}\vec{e}_x + (x + R) \cdot \dot{\Theta} \cdot \vec{e}_\Theta + \dot{y}\vec{e}_y$$

$$\ddot{\vec{r}} = \left(\ddot{x} - (R + x)\dot{\Theta}^2\right)\vec{e}_x + \ddot{y}\vec{e}_y + \left(2\dot{x}\dot{\Theta} + (R + x)\ddot{\Theta}\right)\vec{e}_\Theta$$

$$= \frac{e \cdot q}{m\gamma} \cdot \dot{\vec{r}} \times \vec{B},$$ comparing $\vec{e}_x, \vec{e}_y, \vec{e}_\Theta$ -components:

$$\Longrightarrow \ddot{x} - (R + x)\dot{\Theta}^2 = -\frac{e \cdot q \cdot v \cdot B_y}{m \cdot \gamma} \qquad ①$$

$$\ddot{y} = \frac{e \cdot q \cdot v \cdot B_x}{m \cdot \gamma} \qquad ②$$

$$2\dot{x}\dot{\Theta} + (x + R)\ddot{\Theta} = \frac{e \cdot q}{m\gamma}\left[\dot{x}B_y - \dot{y}B_x\right] \approx 0 \quad \text{since} \quad \boxed{\begin{matrix} \dot{x} \ll v \\ \dot{y} \ll v \end{matrix}, \text{lin.approx.!}}$$

Transverse Dynamics

$\frac{d^2}{dt^2} = v^2 \frac{d^2}{ds^2} = \frac{p^2}{m^2\gamma^2}\frac{d^2}{ds^2}$; ①,② & $\dot{\Theta} = \frac{v}{(R+x)}$:

$$x'' = -\frac{q \cdot e \cdot B_y}{p} + \frac{1}{R+x} \qquad ①$$

$$y'' = \frac{q \cdot e \cdot B_x}{p} = \frac{q \cdot e}{p} G_y \cdot y := k_y \cdot y \qquad ②$$

off momentum $\delta p = p - p_0$

now use: $p \approx p_0 \left[1 + \frac{\delta p}{P_0}\right]$; $\frac{1}{R+x} \approx \frac{1}{R} - \frac{x}{R^2}$

$$B_y = G \quad \cdot x + \frac{p_0}{q \cdot e \cdot R};\; k_x := \frac{e \cdot q \cdot G}{p_0};\; B_x = G \quad \cdot y$$

$$\Longrightarrow \quad x'' + x\left[k_x + \frac{1}{R^2}\right] = \frac{1}{R}\frac{\delta p}{p_0}$$

$$y'' - k_y \cdot y = 0$$

$$G_x = G_y \rightarrow k_x = k_y = k$$

$$x'' + x\left[k + \frac{1}{R^2}\right] = \frac{1}{R}\frac{\delta p}{p_0}$$

$$y'' - k \cdot y = 0$$

foc/defoc in x, defoc/foc in y

$k = k(s)$, $R = R(s)$

"Hill's equation"

Linacs: $R = \infty \rightarrow$ $x'' + k(s)x = 0$, $y'' - k(s)y = 0$ quasi-harmonic oscillator

Envelope Equation

Ansatz: $x(s) = A \cdot u(s) \cdot \cos[\Psi(s) + \Phi_0]$

A & Φ_0 from $x(s=0)$ & $x'(s=0)$

into differential equation for $x(s)$:

$$\Longrightarrow A[u'' - u\Psi'^2 + k \cdot u] \cdot \cos[\Psi + \Phi_0] - A[2u'\Psi' + u\Psi''] \cdot \sin[\Psi + \Phi_0] = 0$$

to be valid $\forall(A, \Phi_0) \rightarrow$ sin & cos - terms must be zero!

$$u'' - u\Psi'^2 + ku = 0 \qquad (*)$$

$$2u'\Psi' + u\Psi'' = 0$$

↳ $$\frac{\Psi''}{\Psi'} = -2\frac{u'}{u}$$

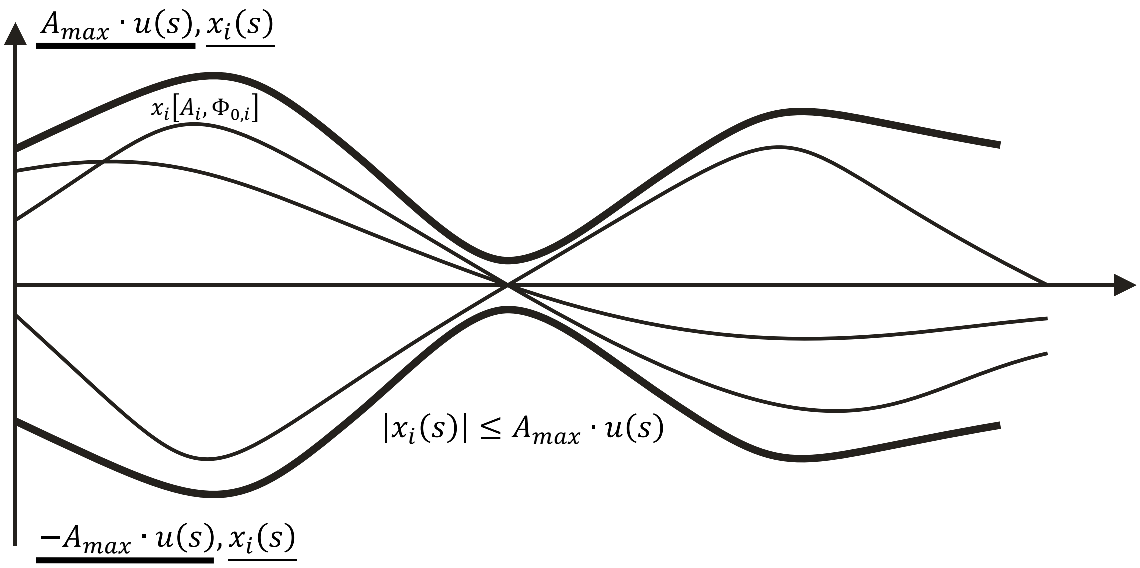
→ $\Psi(s) = C_1 \cdot \int_0^s \frac{1}{u^2} ds + C_2$ $\qquad C_1 = 1,\ C_2 = 0$

phase advance	$\Delta\Psi(s_1 \to s_2) = \int_{s_1}^{s_2} \frac{ds}{u^2}$

$\Psi'\ into (*)$:

$\boxed{u'' - \frac{1}{u^3} + ku = 0}$ $\qquad$ Amplitude/Envelope Equation

$\lim_{u\to 0} u'' \to \pm\infty$ $\qquad$ → $\qquad$ u keeps sign $\forall s$!



phase space ellipse

definitions: $\beta(s) \coloneqq u^2(s)$ $\qquad$ "β-function"

$\varepsilon \coloneqq A^2$ $\qquad$ "single part. emittance"

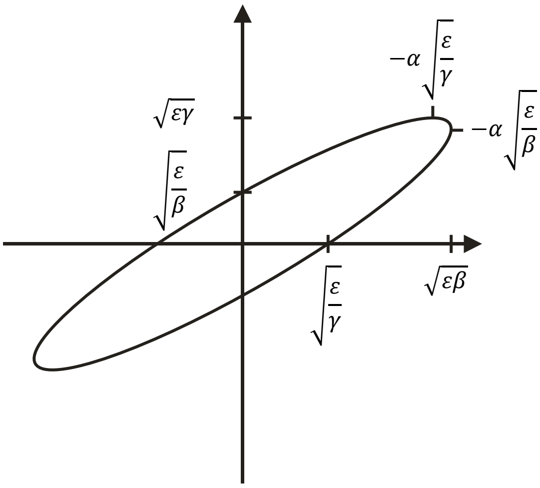
$\alpha(s) \coloneqq -\frac{1}{2}\beta'(s)$ $\qquad$ "α-parameter"

$\gamma(s) \coloneqq \frac{1+\alpha^2(s)}{\beta(s)}$ $\qquad$ "γ-parameter" ≠ rel. !

$$x(s) = \sqrt{\varepsilon \cdot \beta(s)} \cdot \cos[\Psi(s) + \Psi_0]$$

(exercise) → $\gamma(s) \cdot x^2(s) + 2\alpha(s) \cdot x(s) \cdot x'(s) + \beta(s) \cdot x'^2(s) = \varepsilon$

equation of an ellipse in phase space $\begin{bmatrix} x \\ x' \end{bmatrix}$



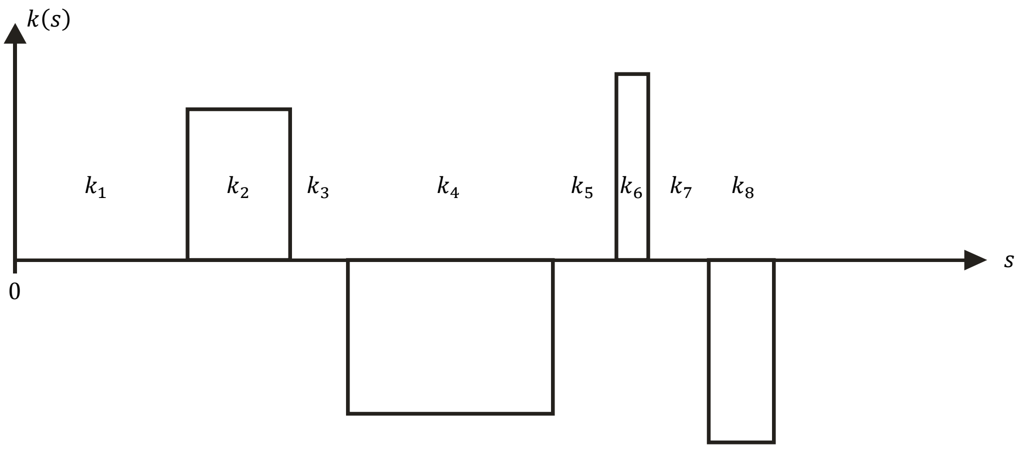
- ε & Φ_0 from initial conditions
- part. with same ε & diff. $\Phi_{0,i}$ remain $\forall s$ on an ellipse
- ellipse area $= \pi \cdot \varepsilon$
- (x, x') area is occupied phase space
- $\beta(s), \alpha(s)$ vary $\rightarrow$ ellipse shape varies, but area $=$ const.

better understanding of $\beta, \alpha, \gamma \rightarrow$ matrix formalism

Matrix Formalism

$$x'' + k(s) \cdot x = 0$$

- general solution too hard
- solve piecewise, since k is from finite elements



Transverse Dynamics

$k = \text{const.} < 0$; K := abs|k|

$C := \cosh[\sqrt{K}s]\,,\ S := \frac{1}{\sqrt{K}} \cdot \sinh[\sqrt{K}s]$

$x = x_0 \cdot C + x_0' \cdot S$

$x' = x_0 \cdot K \cdot S + x_0' \cdot C$

$x'' = x_0 \cdot K \cdot C + x_0' \cdot K \cdot S = Kx$

exp. increase of (x, x')

$k = \text{const.} > 0$

$C := \cos[\sqrt{k}s]\,,\ S := \frac{1}{\sqrt{K}} \cdot \sin[\sqrt{k}s]$

$x = x_0 \cdot C + x_0' \cdot S$

$x' = -x_0 \cdot k \cdot S + x_0' \cdot C$

$x'' = -x_0 \cdot k \cdot C - x_0' \cdot k \cdot S = -kx$

oscillation of (x, x')

$$\begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} C & S \\ C' & S' \end{bmatrix} \begin{bmatrix} x_0 \\ x_0' \end{bmatrix}$$

$$:= R \begin{bmatrix} x_0 \\ x_0' \end{bmatrix}, \ \det\ R = 1$$

S: sin-like solution

C: cos-like solution

(check!)

$$\begin{bmatrix} x \\ x' \end{bmatrix}_s = R[k_n] \cdot R[k_{n-1}] \cdots R[k_2] \cdot R[k_1] \cdot \begin{bmatrix} x_0 \\ x_0' \end{bmatrix}$$

$$= \tilde{R}[k_1 \dots k_n] \begin{bmatrix} x_0 \\ x_0' \end{bmatrix}, \qquad \det \tilde{R} = \det[k_1] \cdot \det[k_2] \cdots \det[k_{n-1}] \cdot \det[k_n]$$

$$= \quad 1 \quad \cdot \quad 1 \quad \cdots \quad 1 \quad \cdot \quad 1 \quad = \quad 1$$

Transport Matrices:

Drift, i.e. $\quad k \equiv 0 \quad$ $\begin{matrix} x'' = 0 \\ y'' = 0 \end{matrix}$ $\quad$ $\begin{matrix} C = 1, & C' = 0 \\ S = s, & S' = 1 \end{matrix}$ $\quad$ $R_{Dx} = R_{Dy} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$

$$R_D = \begin{bmatrix} [R_{Dx}] & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & [R_{Dy}] \end{bmatrix} = \begin{bmatrix} 1 & s & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

quadrupole: $B_x = G \cdot y = \frac{p \cdot k}{q \cdot e} \cdot y\,,\ B_y = G \cdot x = \frac{p \cdot k}{q \cdot e} \cdot x$

$x'' + kx = 0, \qquad y'' - ky = 0, \qquad \Omega := \sqrt{|k|} \cdot s$

Transverse Dynamics

$$R_q(k>0) = \begin{bmatrix} \cos\Omega & \frac{1}{\sqrt{|k|}}\sin\Omega & 0 & 0 \\ -\sqrt{|k|}\sin\Omega & \cos\Omega & 0 & 0 \\ 0 & 0 & \cosh\Omega & \frac{1}{\sqrt{|k|}}\sinh\Omega \\ 0 & 0 & \sqrt{|k|}\sinh\Omega & \cosh\Omega \end{bmatrix}$$

(x, x') oscillation

(y, y') exp.growth

$$R_q(k<0) = \begin{bmatrix} \cosh\Omega & \frac{1}{\sqrt{|k|}}\sinh\Omega & 0 & 0 \\ \sqrt{|k|}\sinh\Omega & \cosh\Omega & 0 & 0 \\ 0 & 0 & \cos\Omega & \frac{1}{\sqrt{|k|}}\sin\Omega \\ 0 & 0 & -\sqrt{|k|}\sin\Omega & \cos\Omega \end{bmatrix}$$

(x, x') exp. growth

(y, y') oscillation

element of length L: $s \to L$

Twiss Parameters

R: Transport-Matrix $\quad \begin{bmatrix} x \\ x' \end{bmatrix}_f = R \begin{bmatrix} x \\ x' \end{bmatrix}_i$

f: "final" ≙ after element
i: "initial" ≙ before element

$\vec{u}_f = R \cdot \vec{u}_i$ here $u \neq$ envelope parameter

$$\left.\begin{matrix} u_{f1} = x_f \\ u_{f2} = x'_f \end{matrix}\right\} \; u_{f1} = R_{11} \cdot u_{i1} + R_{12} \cdot u_{i2} \coloneqq \sum_{j=1}^{2} R_{1j} \cdot u_{ij} \coloneqq R_{1j} \cdot u_{ij}$$

<u>second order beam moments:</u>

- $\langle x\, x \rangle \coloneqq \frac{x_j \cdot x_j}{N}$; square of rms-beam size
- $\langle x\, x' \rangle \coloneqq \frac{x_j \cdot x'_j}{N}$; rms-beam correlation
- $\langle x'\, x' \rangle \coloneqq \frac{x'_j \cdot x'_j}{N}$; square of rms-beam divergence

Transverse Dynamics

<u>transport of 2nd - order moments:</u>

$$\left.\begin{matrix} u_{fa} = R_{an}u_{in} \\ u_{fb} = R_{bm}u_{im} \end{matrix}\right\} \Longrightarrow \langle u_a \cdot u_b \rangle_f = \frac{1}{N}\left[R_{an}R_{bm} \cdot u_{in}u_{im}\right]$$

$$= R_{an}R_{bm}\langle u_n u_m \rangle_i$$

$$= R\begin{bmatrix} \langle u_1u_1\rangle_i & \langle u_1u_2\rangle_i \\ \langle u_2u_1\rangle_i & \langle u_2u_2\rangle_i \end{bmatrix} \cdot R^{\mathrm{T}}$$

$$M_f \coloneqq RM_iR^{\mathrm{T}}$$

$M \coloneqq$ "beam moments matrix"

<u>meaning of β, α, γ:</u>

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon = \begin{bmatrix} x & x' \end{bmatrix}\begin{bmatrix} \gamma & \alpha \\ \alpha & \beta \end{bmatrix}\begin{bmatrix} x \\ x' \end{bmatrix} \coloneqq \vec{u}^{\mathrm{T}} \cdot A^{-1} \cdot \vec{u}$$

$A \coloneqq \begin{bmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{bmatrix}$ $\quad R$: Transport – Matrix $\quad i \xrightarrow{R} f$

$$\vec{u}_f = R \cdot \vec{u}_i\,, \quad \vec{u}_i = R^{-1} \cdot \vec{u}_f$$

$$\vec{u}_i^{\mathrm{T}} \cdot A_i^{-1} \cdot \vec{u}_i = \vec{u}_f^{\mathrm{T}} \cdot A_f^{-1} \cdot \vec{u}_f = \varepsilon = \text{const.}\,!$$

$$\left[R^{-1} \cdot \vec{u}_f\right]^{\mathrm{T}} \cdot A_i^{-1} \cdot R^{-1} \cdot \vec{u}_f = \vec{u}_f^{\mathrm{T}} \cdot A_f^{-1} \cdot \vec{u}_f$$

$$\vec{u}_f^{\mathrm{T}} \cdot \underline{[R^{-1}]^{\mathrm{T}} A_i^{-1} R^{-1}} \cdot \vec{u}_f = \vec{u}_f^{\mathrm{T}} \cdot \underline{A_f^{-1}} \cdot \vec{u}_f$$

$$\Longrightarrow [R^{-1}]^{\mathrm{T}} A_i^{-1} R^{-1} = A_f^{-1}$$

$$A_f = R \cdot A_i \cdot [[R^{-1}]^{\mathrm{T}}]^{-1}$$

$$= R \cdot A_i \cdot R^{\mathrm{T}}, \qquad A \text{ transforms like } M$$

→ β, α, γ transform like $\langle x\, x\rangle$, $\langle x\, x'\rangle$, $\langle x' x'\rangle$

→ define $\quad \beta(s) \coloneqq \frac{\langle x\, x\rangle}{\varepsilon_{rms}}, \quad \alpha(s) \coloneqq -\frac{\langle x\, x'\rangle}{\varepsilon_{rms}}, \quad \gamma(s) \coloneqq \frac{\langle x'\, x'\rangle}{\varepsilon_{rms}}$

$\gamma(s) = \frac{1+\alpha^2}{\beta}; \quad \beta\gamma - \alpha^2 = 1,$ from previous definition of γ(s)

$$\Longrightarrow \varepsilon_{rms}^2 = \langle x\, x\rangle\langle x' x'\rangle - \langle x\, x'\rangle^2$$

$$= \det M$$

$\varepsilon_{rms} \coloneqq$ rms-beam emittance

$\beta(s),\ \alpha(s),\ \gamma(s),\ \varepsilon_{rms}$: Twiss parameters

Transverse Dynamics

$\sqrt{\beta(s) \cdot \varepsilon_{rms}}$ = rms-beam size

$-\alpha(s) \cdot \varepsilon_{rms}$ = rms-beam correlation

$\sqrt{\gamma(s) \cdot \varepsilon_{rms}}$ = rms-beam divergence

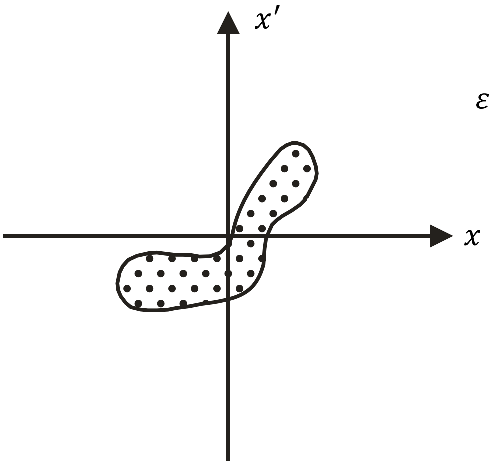
$$\Delta\Psi = \int_{s_1}^{s_2} \frac{ds}{\beta(s)}$$; beam phase advance

Twiss-Parameters calculated from part. coord. in phase space.

single part.emittance of part j: $\varepsilon_j = \gamma x_j^2 + 2\alpha x_j x_j' + \beta {x_j'}^2$

<u>exercise:</u> ε_{rms} invariant under quad & drift transformations.

<u>General Emittance Definition</u>

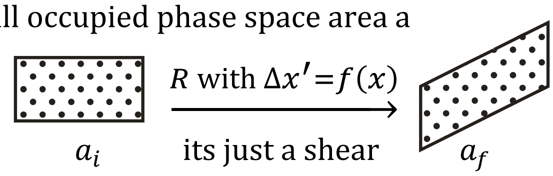


$$\varepsilon = \frac{1}{\pi} \cdot \text{area in phase space occupied by beam}$$

ε is preserved by all elements with

- $\Delta x' = f(x)$: quad, n-pole magnet, rf-gap
- $\Delta x = f(x')$: drift

infinitessimal small occupied phase space area a



$a_i = a_f$!

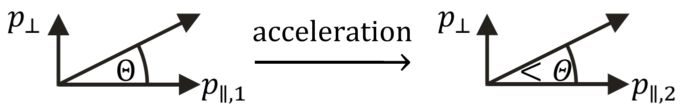
all infinitessimal areas preserved

→ full area is preserved

analogue with $\Delta x = f(x')$ (drift) extension of concept to 6 dimensions is Liouvilles theorem:

6D-Emittance is preserved by actions $\Delta r = f(r')$ & $\Delta r' = f(r)$

acceleration just reduces x':



$$x' = \frac{p_\perp}{p_{\parallel,1}} = \frac{p_\perp}{m \cdot \gamma_1 \cdot \beta_1 \cdot c} \longrightarrow \frac{p_\perp}{m \cdot \gamma_2 \cdot \beta_2 \cdot c}$$

→ acceleration reduces emittance with $(\beta\gamma)^{-1}$

acceleration preserves $\beta\gamma \cdot \varepsilon_{rms,x}$, *called "normalized emittance"*